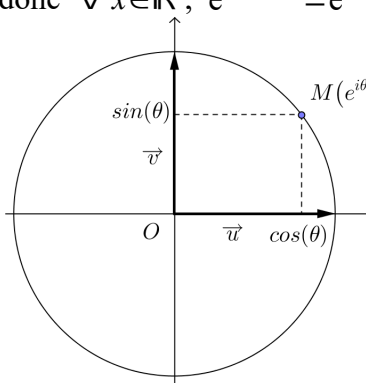
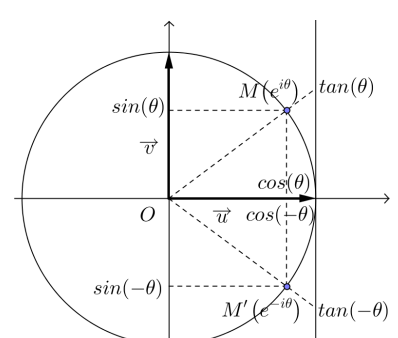
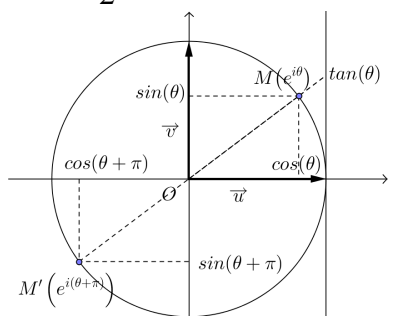
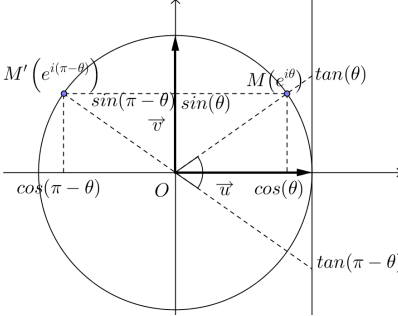
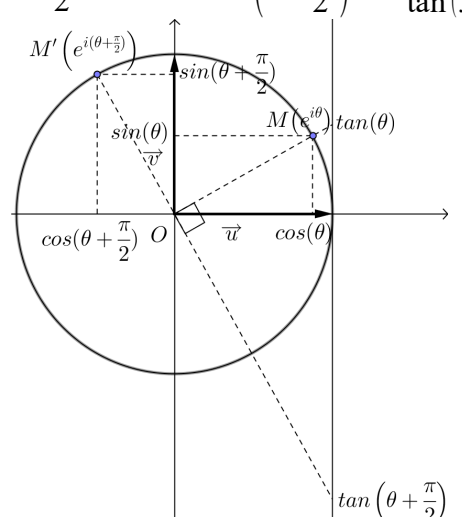
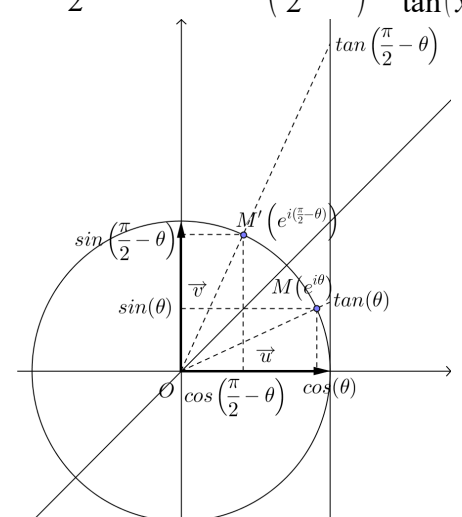


Trigonométrie

Les méthodes pour retrouver ces formules sont plus importantes que les formules elles-mêmes.

I. Opérations sur les arguments

<p>Périodicité :</p> $\forall k \in \mathbb{Z}, e^{i2k\pi} = 1^k = 1 \text{ donc } \forall x \in \mathbb{R}, e^{i(x+2k\pi)} = e^{ix}$ $\begin{cases} \cos(x+2k\pi) = \cos(x) \\ \sin(x+2k\pi) = \sin(x) \end{cases}$ 	<p>Parité :</p> $\forall x \in \mathbb{R}, e^{-ix} = \overline{e^{ix}} \Rightarrow \begin{cases} \cos(-x) = \cos(x) \\ \sin(-x) = -\sin(x) \end{cases}$ <p>Si $x \neq \frac{\pi}{2} \lfloor \pi \rfloor$ alors $\tan(-x) = -\tan(x)$</p> 
<p>$\forall k \in \mathbb{Z}, e^{ik\pi} = (-1)^k \Rightarrow \forall x \in \mathbb{R}, e^{i(x+k\pi)} = (-1)^k e^{ix}$</p> $\begin{cases} \cos(x+k\pi) = (-1)^k \cos(x) \\ \sin(x+k\pi) = (-1)^k \sin(x) \end{cases}$ <p>$\forall k \in \mathbb{Z}$ si $x \neq \frac{\pi}{2} \lfloor \pi \rfloor$ alors $\tan(x+k\pi) = \tan(x)$</p> 	<p>$\forall x \in \mathbb{R}, e^{i(\pi-x)} = -e^{-ix} = -\overline{e^{ix}}$</p> $\begin{cases} \cos(\pi-x) = -\cos(x) \\ \sin(\pi-x) = \sin(x) \end{cases}$ <p>Si $x \neq \frac{\pi}{2} \lfloor \pi \rfloor$ alors $\tan(\pi-x) = -\tan(x)$</p> 
<p>$e^{i\frac{\pi}{2}} = i$ donc $\forall x \in \mathbb{R}, e^{i(x+\frac{\pi}{2})} = i e^{ix}$</p> $\begin{cases} \cos(x+\frac{\pi}{2}) = -\sin(x) \\ \sin(x+\frac{\pi}{2}) = \cos(x) \end{cases}$ <p>Si $x \neq \frac{\pi}{2} \lfloor \pi \rfloor$ alors $\tan(x+\frac{\pi}{2}) = -\frac{1}{\tan(x)}$</p> 	<p>$\forall x \in \mathbb{R}, e^{i(\frac{\pi}{2}-x)} = i \overline{e^{ix}}$</p> $\begin{cases} \cos(\frac{\pi}{2}-x) = \sin(x) \\ \sin(\frac{\pi}{2}-x) = \cos(x) \end{cases}$ <p>Si $x \neq \frac{\pi}{2} \lfloor \pi \rfloor$ alors $\tan(\frac{\pi}{2}-x) = \frac{1}{\tan(x)}$</p> 

Formules d'addition, et de duplication :

$$e^{i(a+b)} = e^{ia} \times e^{ib} \Rightarrow \begin{cases} \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b) \end{cases}$$

$$\begin{aligned} \cos(2a) &= \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 \\ \sin(2a) &= 2\cos(a)\sin(a) \end{aligned}$$

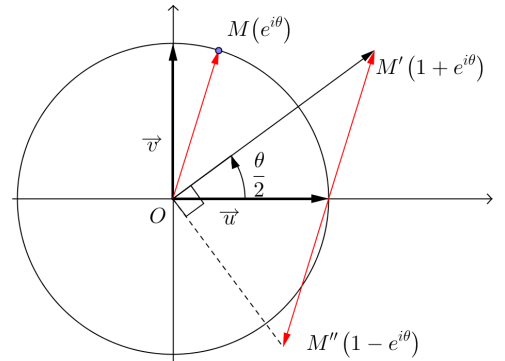
$$\tan(a+b) = \frac{\frac{\cos(a)\sin(b) + \sin(a)\cos(b)}{\cos(a)\cos(b)}}{\frac{\cos(a)\cos(b) - \sin(a)\sin(b)}{\cos(a)\cos(b)}} = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

$$\tan(2a) = \frac{2\tan(a)}{1 - \tan^2(a)}$$

Formules de l'angle moitié :

$$1 + e^{i\theta} = e^{i\frac{\theta}{2}} \left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}} \right) = 2\cos\left(\frac{\theta}{2}\right) e^{i\frac{\theta}{2}} \Rightarrow \begin{cases} 1 + \cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) \\ \sin(\theta) = 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \end{cases}$$

$$1 - e^{i\theta} = e^{i\frac{\theta}{2}} \left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) = -2i\sin\left(\frac{\theta}{2}\right) e^{i\frac{\theta}{2}} \Rightarrow \begin{cases} 1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right) \\ \sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \end{cases}$$



Formules de « la tangente de l'angle moitié » : $e^{i\theta} = \frac{e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}} = \frac{\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)} = \frac{1 + i\tan\left(\frac{\theta}{2}\right)}{1 - i\tan\left(\frac{\theta}{2}\right)}$ si $\frac{\theta}{2} \neq \frac{\pi}{2} [\pi]$

$$\cos(\theta) = \frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} \quad \text{et} \quad \sin(\theta) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} \quad \text{et} \quad \tan(\theta) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$$

Développements : Formule de Moivre et du binôme de Newton

$$e^{in\theta} = (\cos(\theta) + i\sin(\theta))^n = \sum_{k=0}^n (i)^{n-k} \binom{n}{k} \cos^k(\theta) \sin^{n-k}(\theta) \text{ ainsi } \begin{cases} \cos(n\theta) = \Re\left((\cos(\theta) + i\sin(\theta))^n\right) \\ \sin(n\theta) = \Im\left((\cos(\theta) + i\sin(\theta))^n\right) \end{cases}$$

II. Opérations sur les fonctions trigonométriques

Liens entre les fonctions trigonométriques :

$$\forall \theta \in \mathbb{R}, \cos^2\theta + \sin^2\theta = 1 \text{ et si } \theta \neq \frac{\pi}{2} [\pi] \text{ alors } \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \text{ donc } \tan^2\theta = \frac{1 - \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} - 1$$

Formules de linéarisation : Formules d'Euler

$$\cos(a)\cos(b) = \frac{e^{ia} + e^{-ia}}{2} \times \frac{e^{ib} + e^{-ib}}{2} = \frac{e^{i(a+b)} + e^{-i(a+b)} + e^{i(a-b)} + e^{-i(a-b)}}{4} = \frac{1}{2} (\cos(a+b) + \cos(a-b))$$

$$\cos(a)\sin(b) = \frac{e^{ia} + e^{-ia}}{2} \times \frac{e^{ib} - e^{-ib}}{2i} = \frac{e^{i(a+b)} - e^{-i(a+b)} + e^{i(b-a)} - e^{-i(b-a)}}{4i} = \frac{1}{2} (\sin(a+b) + \sin(b-a))$$

$$\sin(a)\sin(b) = \frac{e^{ia} - e^{-ia}}{2i} \times \frac{e^{ib} - e^{-ib}}{2i} = -\frac{e^{i(a+b)} + e^{-i(a+b)} - e^{i(a-b)} - e^{-i(a-b)}}{4} = \frac{1}{2} (\cos(a-b) - \cos(a+b))$$

$$\cos(\theta)^n = \frac{e^{i\theta} + e^{-i\theta}}{2} \Rightarrow (\cos(\theta))^n = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} e^{ik\theta} e^{-i(n-k)\theta} = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} e^{i(2k-n)\theta}$$

$$(\cos(\theta))^n \text{ est combinaison linéaire de } \cos(n\theta), \cos((n-2)\theta), \dots \begin{cases} \cos(\theta) \text{ si } n \text{ est impaire} \\ 1 \text{ si } n \text{ est paire} \end{cases}$$

$$\sin(\theta)^n = \frac{e^{i\theta} - e^{-i\theta}}{2i} \Rightarrow (\sin(\theta))^n = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^n = \frac{1}{(2i)^n} \sum_{k=0}^n \binom{n}{k} e^{ik\theta} (-1)^{n-k} e^{-i(n-k)\theta} = \frac{1}{(2i)^n} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} e^{i(2k-n)\theta}$$

$$(\sin(\theta))^n \text{ est combinaison linéaire de } \begin{cases} \text{si } n \text{ est pair : } \cos(n\theta); \cos((n-2)\theta); \cos((n-4)\theta); \dots; 1 \\ \text{si } n \text{ est impaire : } \sin(n\theta); \sin((n-2)\theta); \sin((n-4)\theta); \dots; \sin(\theta) \end{cases}$$

$$\text{Cas particuliers : } \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}; \quad \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \quad \text{et} \quad \tan^2(\theta) = \frac{1 - 2\cos(\theta)}{1 + 2\cos(\theta)}$$

Formules de factorisation

$a \cos(\theta) + b \sin(\theta) = \Re(e^{i\theta}(a - ib))$ ainsi, pour $a - ib = \rho e^{i\alpha}$: $a \cos(\theta) + b \sin(\theta) = \Re(\rho e^{i(\theta+\alpha)}) = \rho \cos(\theta + \alpha)$

$$e^{ia} + e^{ib} = e^{\frac{ia}{2}} e^{\frac{ib}{2}} \left(e^{\frac{i(a-b)}{2}} + e^{-\frac{i(a-b)}{2}} \right) = e^{\frac{i(a+b)}{2}} 2 \cos\left(\frac{a-b}{2}\right) \Rightarrow \begin{cases} \cos(a) + \cos(b) = 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) \\ \sin(a) + \sin(b) = 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) \end{cases}$$
$$e^{ia} - e^{ib} = e^{\frac{ia}{2}} e^{\frac{ib}{2}} \left(e^{\frac{i(a-b)}{2}} - e^{-\frac{i(a-b)}{2}} \right) = e^{\frac{i(a+b)}{2}} 2i \sin\left(\frac{a-b}{2}\right) \Rightarrow \begin{cases} \cos(a) - \cos(b) = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \\ \sin(a) - \sin(b) = 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \end{cases}$$

si $\theta \neq 0 \pmod{2\pi}$ alors : $\sum_{k=0}^n e^{ik\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{e^{i\left(\frac{n+1}{2}\theta\right)} \left(e^{-i\left(\frac{n+1}{2}\theta\right)} - e^{i\left(\frac{n+1}{2}\theta\right)} \right)}{e^{i\left(\frac{\theta}{2}\right)} \left(e^{-i\left(\frac{\theta}{2}\right)} - e^{i\left(\frac{\theta}{2}\right)} \right)} = e^{i\left(\frac{n}{2}\theta\right)} \frac{-2i \sin\left(\frac{n+1}{2}\theta\right)}{-2i \sin\left(\frac{\theta}{2}\right)}$

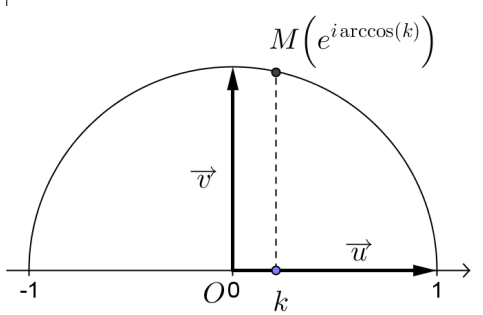
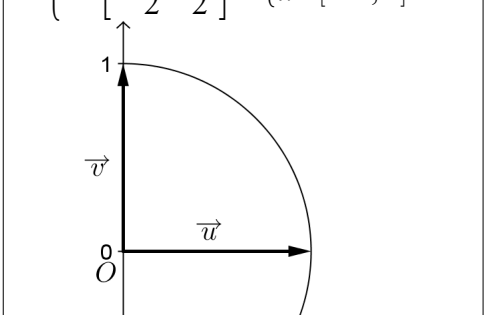
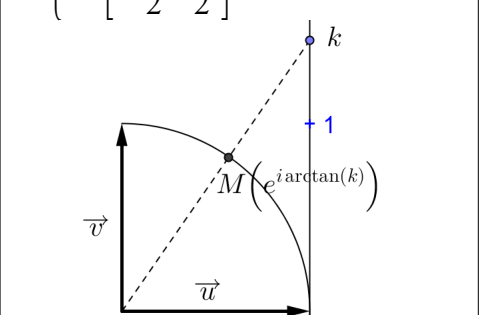
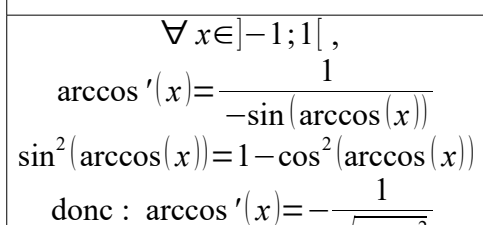
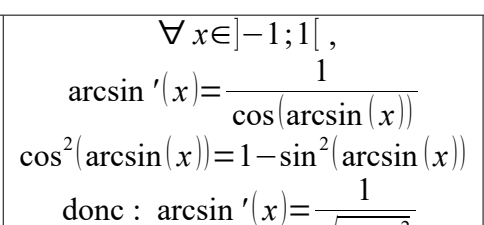
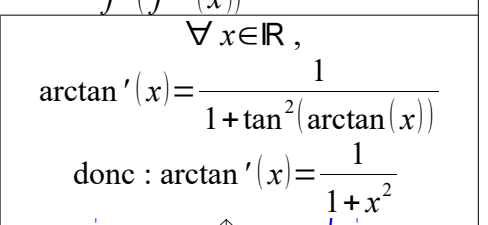
Donc $\sum_{k=0}^n \cos(k\theta) = \frac{\cos\left(\frac{n}{2}\theta\right) \sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$ et $\sum_{k=0}^n \sin(k\theta) = \frac{\sin\left(\frac{n}{2}\theta\right) \sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$

Si $a \neq \frac{\pi}{2} \pmod{\pi}$ et $b \neq \frac{\pi}{2} \pmod{\pi}$ alors $\tan(a) + \tan(b) = \frac{\sin(a)}{\cos(a)} + \frac{\sin(b)}{\cos(b)} = \frac{\sin(a)\cos(b) + \sin(b)\cos(a)}{\cos(a)\cos(b)} = \frac{\sin(a+b)}{\cos(a)\cos(b)}$

Dérivées : $\forall x \in \mathbb{R}$, $(e^{ix})' = i e^{ix} = e^{\frac{i\pi}{2}} e^{ix} = e^{i\left(x + \frac{\pi}{2}\right)} \Rightarrow \begin{cases} \cos'(x) = -\sin(x) = \cos\left(x + \frac{\pi}{2}\right) \\ \sin'(x) = \cos(x) = \cos\left(x + \frac{\pi}{2}\right) \end{cases}$

$\forall x \in \left] -\frac{\pi}{2}; \frac{\pi}{2} \right[$, $\tan'(x) = \left(\frac{\sin(x)}{\cos(x)} \right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \tan^2 x$

III. Équations trigonométriques

$\cos(a) = \cos(b) \Leftrightarrow a = \pm b + 2\pi$ <p> $[0; \pi] \rightarrow [-1; 1]$ $\theta \mapsto \cos(\theta)$ $\arccos(k) \leftarrow k$ $\begin{cases} x = \cos(\theta) \\ \theta \in [0; \pi] \end{cases} \Leftrightarrow \begin{cases} \arccos(x) = \theta \\ x \in [-1; 1] \end{cases}$ </p> 	$\sin(a) = \sin(b) \Leftrightarrow (a = b + 2\pi) \text{ ou } a = \pi - b + 2\pi$ <p> $[-\frac{\pi}{2}; \frac{\pi}{2}] \rightarrow [-1; 1]$ $\theta \mapsto \sin(\theta)$ $\arcsin(k) \leftarrow k$ $\begin{cases} x = \sin(\theta) \\ \theta \in [-\frac{\pi}{2}; \frac{\pi}{2}] \end{cases} \Leftrightarrow \begin{cases} \arcsin(x) = \theta \\ x \in [-1; 1] \end{cases}$ </p> 	$\tan(a) = \tan(b) \Leftrightarrow a = b + \pi$ <p> $]-\frac{\pi}{2}; \frac{\pi}{2}[\rightarrow \mathbb{R}$ $\theta \mapsto \tan(\theta)$ $\arctan(k) \leftarrow k$ $\begin{cases} x = \tan(\theta) \\ \theta \in]-\frac{\pi}{2}; \frac{\pi}{2}[\end{cases} \Leftrightarrow \arctan(x) = \theta$ </p> 
$\forall x \in [-1; 1], \cos(\arccos(x)) = x$ Soit $x \in [(2k-1)\pi; (2k+1)\pi]$ avec $k \in \mathbb{Z}$ alors $\arccos(\cos(x)) = x - 2k\pi $	$\forall x \in [-1; 1], \sin(\arcsin(x)) = x$ Soit $x \in \mathbb{R}$, si $x \in [-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi]$ avec $k \in \mathbb{Z}$ alors $\arcsin(\sin(x)) = (-1)^k (x - k\pi)$	$\forall x \in \mathbb{R}, \tan(\arctan(x)) = x$ Soit $x \in \mathbb{R}$, si $x \in]-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi[$ avec $k \in \mathbb{Z}$ alors $\arctan(\tan(x)) = x - k\pi$
$\forall x \in I, f \circ f^{-1}(x) = x \Rightarrow \text{si } f'(f^{-1}(x)) \neq 0 \text{ alors } (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$		
$\forall x \in]-1; 1[, \arccos'(x) = \frac{1}{-\sin(\arccos(x))}$ $\sin^2(\arccos(x)) = 1 - \cos^2(\arccos(x))$ donc : $\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$ 	$\forall x \in]-1; 1[, \arcsin'(x) = \frac{1}{\cos(\arcsin(x))}$ $\cos^2(\arcsin(x)) = 1 - \sin^2(\arcsin(x))$ donc : $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$ 	$\forall x \in \mathbb{R}, \arctan'(x) = \frac{1}{1 + \tan^2(\arctan(x))}$ donc : $\arctan'(x) = \frac{1}{1+x^2}$ 

$$\forall x \in [-1; 1], \cos(\arccos(x)) = \sin(\arcsin(x)) \text{ or } \forall \theta \in \mathbb{R}, \sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\text{Donc } \cos(\arccos(x)) = \cos\left(\frac{\pi}{2} - \arcsin(x)\right) \text{ or } \begin{cases} \arccos(x) \in [0; \pi] \\ \frac{\pi}{2} - \arcsin(x) \in [0; \pi] \end{cases}$$

$$\text{Ainsi } \arccos(x) = \frac{\pi}{2} - \arcsin(x)$$